CAMBRIDGE ASSESSMENT

STEP Solutions 2009

Mathematics

STEP 9465, 9470, 9475

UNIVERSITY of CAMBRIDGE
Local Examinations Syndicate

STEP III, Solutions 2009

Section A: Pure Mathematics

1. The result for p can be found via calculating the equation of the line SV ($y - ms = \frac{ms - nv}{s - v}(x - s)$) or similar triangles. The result for q follows from that for p (given in the question) by suitable interchange of letters to give
$q = \frac{(m-n)tu}{mt - nu}$

As S and T lie on the circle, s and t are solutions of the equation
$\lambda^2 + (m\lambda - c)^2 = r^2$ i.e. $\left(1 + m^2\right)\lambda^2 - 2mc\lambda + \left(c^2 - r^2\right) = 0$

and so from considering sum and product of roots, $st = \frac{c^2 - r^2}{1 + m^2}$, and $s + t = \frac{2mc}{1 + m^2}$

Similarly $uv = \frac{c^2 - r^2}{1 + n^2}$, and $u + v = \frac{2nc}{1 + n^2}$ can be deduced by interchanging letters.

Substituting from the earlier results $p + q = \frac{(m-n)sv}{ms - nv} + \frac{(m-n)tu}{mt - nu}$ which can be simplified to $\frac{(m-n)}{(ms - nv)(mt - nu)}\left(stm(u + v) - nuv(s + t)\right)$
and then substituting the sum and product results yields the required result.

2 (i) The five required results are straightforward to write down, merely observing that initial terms in the summations are zero.

(ii) Substituting the series from (i) in the differential equation yields that
$-a_1 + 3a_3x^2 + \left(8a_4 + 4a_0\right)x^3 + \ldots.. = 0$, after having collected like terms.
Thus, comparing constants and x^2 coefficients $a_1 = 0$ and $a_3 = 0$
Comparing coefficients of x^{n-1}, for $n \geq 4$, $n(n-1)a_n - na_n + 4a_{n-4} = 0$ which gives the required result upon rearrangement.

With $a_0 = 1$, $a_2 = 0$, and as $a_1 = 0$, and $a_3 = 0$, we find $a_4 = \frac{-1}{2!}$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{1}{4!}$, etc.

Thus $y = 1 - \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{4!}\left(x^2\right)^4 - \frac{1}{6!}\left(x^2\right)^6 + \ldots.. = \cos\left(x^2\right)$

With $a_0 = 0$, $a_2 = 1$, $y = \left(x^2\right) - \frac{1}{3!}\left(x^2\right)^3 + \frac{1}{5!}\left(x^2\right)^5 - \frac{1}{7!}\left(x^2\right)^7 + \ldots.. = \sin\left(x^2\right)$

3. (i) Substituting the power series and tidying up the algebra yields

$f(t) = \dfrac{1}{\left(1 + \dfrac{t}{2!} + ...\right)}$ and so $\lim\limits_{t \to 0} f(t) = 1$.

Similarly, $f'(t) = \dfrac{(e^t - 1) - te^t}{(e^t - 1)^2} = \dfrac{-\dfrac{1}{2} - t\left(\dfrac{1}{2!} - \dfrac{1}{3!}\right) - .....}{\left(1 + \dfrac{t}{2!} + ...\right)^2}$ and so $\lim\limits_{t \to 0} f'(t) = \dfrac{-1}{2}$

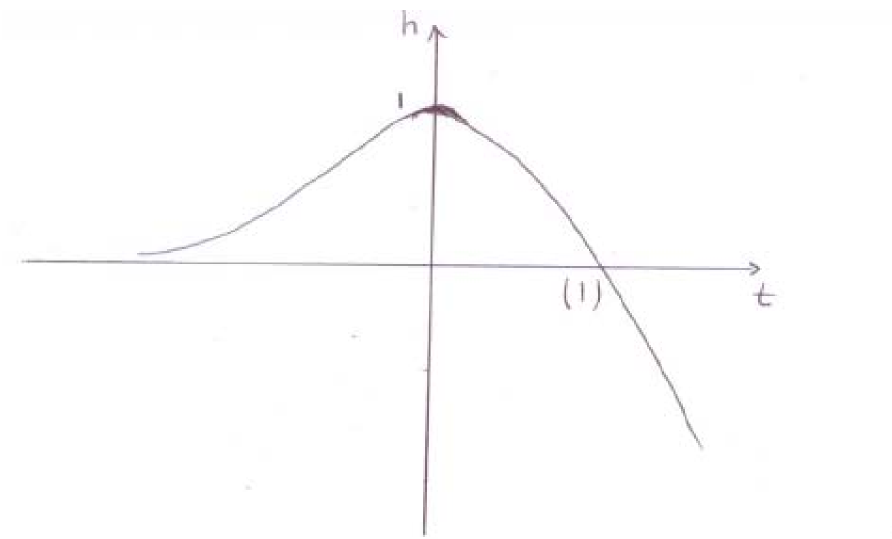
(Alternatively, this can be obtained by de l'Hopital.)

(ii) If we let $g(t) = f(t) + \dfrac{1}{2}t$, then simplifying the algebra gives

$g(t) = \dfrac{t(e^t + 1)}{2(e^t - 1)}$

after which it is can be shown by substituting $-t$ for t
that $g(-t)$ is the same expression.

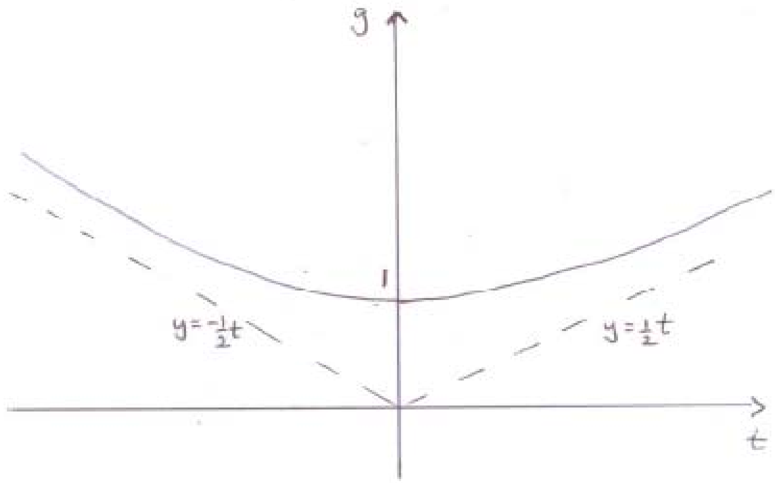
(iii) If we let $h(t) = e^t(1 - t)$, and find its stationary point, sketching the graph gives



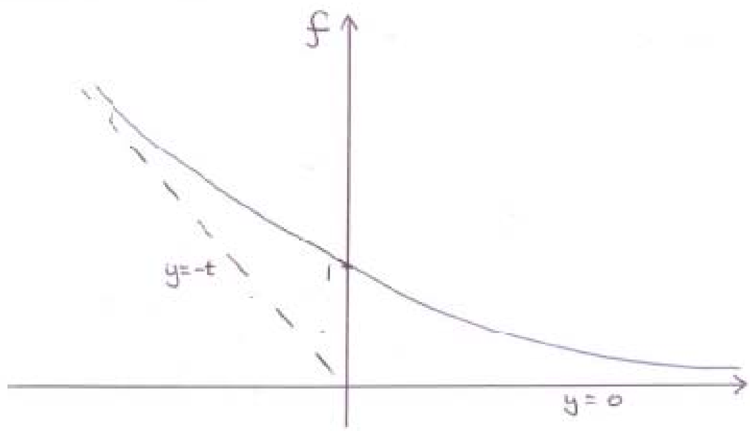
Hence $e^t(1 - t) \leq 1$ and so $e^t(1 - t) - 1 \leq 0$. (Alternatively, a sketch with e^t and $\frac{1}{1-t}$ will yield the result.)

Thus $f'(t) = \frac{(1-t)e^t - 1}{(e^t - 1)^2} \leq 0$, with equality only possible for $t = 0$, but we know $\lim\limits_{t \to 0} f'(t) = \dfrac{-1}{2}$ and so, in fact, *f(t)* is always decreasing i.e. has no turning points.

Considering the graph of $g(t) = f(t) + \frac{1}{2}t$. It passes through (0,1), is symmetrical and approaches $y = \frac{1}{2}t$ as $t \to \infty$ and thus is



Therefore the graph of $f(t) = g(t) - \frac{1}{2}t$ also passes through (0,1), and has asymptotes $y = 0$ and $y = -t$ and thus is



4. (i) Substituting into the definition yields the Laplace transform as

$$\int_0^\infty e^{-st}e^{-bt}f(t)dt = \int_0^\infty e^{-t(s+b)}f(t)dt = F(s+b)$$

(ii) Similarly, a change of variable in the integral using $\mathrm{u} = at$ yields the result.

(iii) Integrating by parts yields this answer.

(iv) A repeated integration by parts obtains

$$F(s) = 1 - s^2F(s)$$

which leads to the stated result.

Using the results obtained in the question, the transform of $\cos qt$ is

$q^{-1}\left(\dfrac{s/q}{s^2/q^2+1}\right)=\dfrac{s}{s^2+q^2}$,and so the transform of $e^{-pt}\cos qt$ is $\dfrac{(s+p)}{(s+p)^2+q^2}$

5. The first result may be obtained by considering

$$(x+y+z)^2-(x^2+y^2+z^2)=2(yz+zx+xy)\quad ,$$

the second by

$$(x^2+y^2+z^2)(x+y+z)=x^3+y^3+z^3+(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)$$

and the third by

$$(x+y+z)^3=(x^3+y^3+z^3)+3(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)+6xyz$$

Considering sums and products of roots, we can deduce that x satisfies the cubic equation $x^3-x^2-\frac{1}{2}x-\frac{1}{6}=0$, as do y and z by symmetry. Multiplying by x^{n-2} , $x^{n+1}=x^n+\frac{1}{2}x^{n-1}+\frac{1}{6}x^{n-2}$, with similar results for y and z. Summing these yields

$$S_{n+1}=S_n+\tfrac{1}{2}S_{n-1}+\tfrac{1}{6}S_{n-2}$$

Alternatively,

$x^{n+1}+y^{n+1}+z^{n+1}=(x+y+z)(x^n+y^n+z^n)-(xy^n+xz^n+yx^n+yz^n+zx^n+zy^n)$

$=1.S_n-(xy+yz+zx)(x^{n-1}+y^{n-1}+z^{n-1})+xyz(x^{n-2}+y^{n-2}+z^{n-2})$

to give the result.

6. Using Euler, $e^{i\beta}-e^{i\alpha}=(\cos\beta-\cos\alpha)+i(\sin\beta-\sin\alpha)$

and so

$$\left|e^{i\beta}-e^{i\alpha}\right|^2=(\cos\beta-\cos\alpha)^2+(\sin\beta-\sin\alpha)^2$$

which can be expanded, and then using Pythagoras, compound and half angle formulae this becomes

$$4\sin^2\frac{1}{2}(\beta-\alpha)$$

$\left|e^{i\beta}-e^{i\alpha}\right|=2\sin\dfrac{1}{2}(\beta-\alpha)$ as both expressions are positive.

Alternative methods employ the factor formulae.

$$\left|e^{i\alpha}-e^{i\beta}\right|\left|e^{i\gamma}-e^{i\delta}\right|+\left|e^{i\beta}-e^{i\gamma}\right|\left|e^{i\alpha}-e^{i\delta}\right|$$

$$=2\sin\left(\frac{1}{2}(\alpha-\beta)\right)2\sin\left(\frac{1}{2}(\gamma-\delta)\right)+2\sin\left(\frac{1}{2}(\beta-\gamma)\right)2\sin\left(\frac{1}{2}(\alpha-\delta)\right)$$

which by use of the factor formulae and cancelling terms may be written

$$2\left(\cos\left(\frac{1}{2}(\alpha-\beta-\gamma+\delta)\right)-\cos\left(\frac{1}{2}(\beta-\gamma+\alpha-\delta)\right)\right)$$

and then again by factor formulae,

$$2\sin\left(\frac{1}{2}(\alpha-\gamma)\right)2\sin\left(\frac{1}{2}(\beta-\delta)\right)$$

which is

$\left|e^{i\alpha}-e^{i\gamma}\right|\left|e^{i\beta}-e^{i\delta}\right|$ as required.

Thus, the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite pairs of sides **(** Ptolemy’s Theorem).

7. (i) This result is simply obtained using the principle of mathematical induction. The $n = 1$ case can be established merely by obtaining f_1 and f_2 from the definition, and then substituting these along with f_0.

(ii)

$$P_0(x) = (1+x^2)\frac{1}{1+x^2} = 1$$

$$P_1(x) = (1+x^2)^2\frac{-2x}{(1+x^2)^2} = -2x$$

$$P_2(x) = (1+x^2)^3\frac{6x^2-2}{(1+x^2)^3} = 6x^2-2$$

$$P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x)$$

which differentiating P_n by the product rule and substituting

$$= (1+x^2)^{n+2} f_{n+1}(x) - (1+x^2)\left((1+x^2)^{n+1} f_{n+1}(x) + (n+1)2x(1+x^2)^n f_n(x)\right) + 2(n+1)x(1+x^2)^{n+1} f_n(x)$$

which is zero.

Again using the principle of mathematical induction and the result just obtained, it can be found that $P_{k+1}(x)$ is a polynomial of degree <u>not greater than</u> $k+1$.

Further, assuming that $P_k(x)$ has term of highest degree, $(-1)^k(k+1)!\,x^k$, as $P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0$, the term of highest degree of $P_{k+1}(x)$ is

$(-1)^k(k+1)!\,kx^{k-1}x^2 - 2(k+1)x(-1)^k(k+1)!\,x^k$

$= (-1)^{k+1}(k+2)!\,x^{k+1}$ as required.

(The form of the term need not be determined, but it must be shown to be non-zero.)

8. (i) Letting $x = e^{-t}$,

$\lim_{x\to 0}[x^m(\ln x)^n] = \lim_{t\to\infty}[(e^{-t})^m(-t)^n] = (-1)^n \lim_{x\to 0}[e^{-mt}t^n] = 0$

and so letting $m = n = 1$, $\lim_{x\to 0}[x\ln x] = 0$.

Thus, $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1$

(ii) Integrating by parts,

$$I_{n+1} = \int_0^1 x^m (\ln x)^{n+1} dx = \left[\frac{x^{m+1}(\ln x)^{n+1}}{m+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{(n+1)(\ln x)^n}{x} dx$$

$$= 0 - 0\ (using\ the\ first\ result) - \int_0^1 \frac{n+1}{m+1} x^m (\ln x)^n dx = -\frac{n+1}{m+1} I_n$$

So $I_n = \frac{-n}{m+1} \times \frac{-(n-1)}{m+1} \times \frac{-(n-2)}{m+1} \times \ldots \times \frac{-1}{m+1} I_0 = \frac{(-1)^n n!}{(m+1)^n} \int_0^1 x^m dx$

$= \frac{(-1)^n n!}{(m+1)^{n+1}}$

(iii) $\int_0^1 x^x dx = \int_0^1 e^{x \ln x} dx = \int_0^1 1 + x \ln x + \frac{x^2 (\ln x)^2}{2!} + \ldots .. dx$

$= 1 + I_1 + \frac{1}{2!} I_2 + \ldots .. \quad = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \ldots ..$ as required.

Section B: Mechanics

9. (i) With V as the speed of projection from P, x and y the horizontal and vertical displacements from P at a time t after projection, and T the time of flight from P to Q, then
$x = Vt \cos\theta$, $y = Vt \sin\theta - \frac{1}{2} g t^2$, $\dot{x} = V \cos\theta$, and $\dot{y} = V\ sin\ \theta - gt$

So $\tan\alpha = \frac{VT \tan\theta - \frac{1}{2} g T^2}{VT \cos\theta} = \tan\theta - \frac{gT}{2V\cos\theta}$, and $\tan\varphi = \frac{V\sin\theta - gT}{V\cos\theta} = \tan\theta - \frac{gT}{V\cos\theta}$

Thus $\tan\theta + \tan\varphi = 2\tan\theta - \frac{gT}{V\cos\theta} = 2\tan\alpha$

(ii) Using the trajectory equation written as a quadratic equation in $\tan\theta$,

$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(\frac{gx^2}{2V^2} + y\right) = 0$, giving $\tan\theta + \tan\theta' = \frac{2V^2}{gx}$, and
$\tan\theta\tan\theta' = 1 + \frac{2V^2 y}{gx^2} = 1 + \frac{2V^2}{gx}\tan\alpha$.

Applying the compound angle formula and substituting, $\tan(\theta + \theta') = -\cot\alpha$
So, $+\theta' = \frac{\pi}{2} + \alpha + n\pi$, and as $0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $0 < \alpha < \frac{\pi}{2}$,
$\theta + \theta' = \frac{\pi}{2} + \alpha$.
Reversing the motion we have, $(-\varphi) + (-\varphi') = \frac{\pi}{2} + (-\alpha) + n'\pi$, and therefore,
$\varphi + \varphi' = \alpha + \left(-n' - \frac{1}{2}\right)\pi = \theta + \theta' - n''\pi$
$0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi' < \frac{\pi}{2}$, and $\varphi < \theta$, $\varphi' < \theta'$
so $\varphi + \varphi' = \theta + \theta' - \pi$, or as required $\theta + \theta' = \varphi + \varphi' + \pi$

10. Supposing that the particle P has mass m, the spring has natural length l, and modulus of elasticity λ, $mg = \frac{\lambda d}{l}$

If the speed of P when it hits the top of the spring is v, then $v = \sqrt{2gh}$

By Newton's second law, the second-order differential equation is thus

$m\ddot{x} = mg - \frac{\lambda x}{l} = mg - \frac{mgx}{d}$ and so $\ddot{x} = g - \frac{gx}{d}$ with initial conditions that $x = 0$, $\dot{x} = \sqrt{2gh}$, when $t = 0$.

$\ddot{x} + \frac{gx}{d} = g$ has complementary function $x = B\cos\omega t + C\sin\omega t$

where $\omega = \sqrt{\frac{g}{d}}$, and particular integral $x = A$, where $A = d$.

The initial conditions yield, $B = -d$ and $C = \sqrt{2dh}$

So $x = d - d\cos\sqrt{\frac{g}{d}}t + \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$.

$d\cos\sqrt{\frac{g}{d}}t - \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$ may be expressed in the form $R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$ where

$R^2 = d^2 + 2dh$, and $\tan\alpha = \frac{\sqrt{2dh}}{d} = \sqrt{\frac{2h}{d}}$

So $x = d - R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$

$x = 0$ next when $t = T$, that is when $2\pi - \left(\sqrt{\frac{g}{d}}T + \alpha\right) = \alpha$

So $\sqrt{\frac{g}{d}}T = 2\pi - 2\alpha = 2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}$ and $T = \sqrt{\frac{d}{g}}\left(2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}\right)$.

11. (i) Conserving momentum yields $MV = M(1 + bx)v$ and so $V = (1 + bx)v$ Written as $V = (1 + bx)\frac{dx}{dt}$, separating variables and integrating

$Vt + c = x + \frac{1}{2}bx^2$, but as $= 0$, when $t = 0$, $c = 0$

So $\frac{1}{2}bx^2 + x - Vt = 0$, and so $x = \frac{-1 \pm \sqrt{1 + 2bVt}}{b}$, except $x > 0$, and thus

$x = \frac{-1 + \sqrt{1 + 2bVt}}{b}$

(ii) $Mf = \frac{d}{dt}(mv) = \frac{d}{dt}(M(1 + bx)v)$

So, $ft + c' = (1 + bx)v$ and as $= 0$, $x = 0$, and $v = V$ we have $c' = V$.

Thus $v = \frac{ft + V}{1 + bx}$ as required.

Separating variables and integrating

$\frac{1}{2}ft^2 + Vt + c'' = x + \frac{1}{2}bx^2$ and as $x = 0$, when $t = 0$, $c'' = 0$

So $\frac{1}{2}bx^2 + x - \frac{1}{2}ft^2 - Vt = 0$, and so $= \frac{-1 \pm \sqrt{1 + fbt^2 + 2bVt}}{b}$, except $x > 0$, and thus

$x = \frac{-1 + \sqrt{1 + fbt^2 + 2bVt}}{b}$

If $1+fbt^2+2bVt$ is a perfect square, then *x* will be linear in *t* and $\frac{dx}{dt}$ will be constant, i.e. if $4b^2V^2-4fb=0$, that is $bV^2=f$
(in which case $x=\frac{-1+\sqrt{1+b^2V^2t^2+2bVt}}{b}=\frac{-1+(1+bVt)}{b}=Vt$, and $v=V$ as expected.)

Otherwise, $=\frac{ft+V}{1+bx}=\frac{ft+V}{\sqrt{1+fbt^2+2bVt}}=\frac{f+\frac{V}{t}}{\sqrt{fb+\frac{2bV}{t}+\frac{1}{t^2}}}$, and as $t\to\infty$, $v\to\frac{f}{\sqrt{fb}}=\sqrt{\frac{f}{b}}$, a constant, as required.

Section C: Probability and Statistics

12. (i) $E(X_1)=\frac{1}{2}k$, $E(X_2|X_1=x_1)=\frac{1}{2}x_1$, and so
$E(X_2)=\sum\frac{1}{2}x_1\,P(X_1=x_1)=\frac{1}{2}E(X_1)=\frac{1}{4}k$
$\sum_{i=1}^{\infty}E(X_i)=\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i k=k$ using the sum of an infinite GP.

(ii) $G_Y(t)=E(t^Y)=E\left(t^{\sum_{i=1}^{k}Y_i}\right)=\prod_{i=1}^{k}E(t^{Y_i})$
$P(Y_i=0)=\frac{1}{2}$, $(Y_i=1)=\frac{1}{4}$, … , $P(Y_i=r)=\left(\frac{1}{2}\right)^{r-1}$
and so $E(t^{Y_i})=\frac{1}{2}+\frac{1}{4}t+\cdots+\left(\frac{1}{2}\right)^{r-1}t^r+\ ...=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}t\right)}=\frac{1}{2-t}$ (infinite GP)
Thus $G_Y(t)=\prod_{i=1}^{k}\frac{1}{2-t}=\left(\frac{1}{2-t}\right)^k$

$G'_Y(t)=\frac{k}{(2-t)^{k+1}}$, $G''_Y(t)=\frac{k(k+1)}{(2-t)^{k+2}}$, and $G^{(r)}{}_Y(t)=\frac{k(k+1)(k+2)...(k+r-1)}{(2-t)^{k+r}}$
and so $E(Y)=G'_Y(1)=k$, $Var(Y)=G''_Y(1)+G'_Y(1)-\left(G'_Y(1)\right)^2=2k$
and $P(Y=r)=\frac{G^{(r)}{}_Y(0)}{r!}=\frac{k(k+1)(k+2)...(k+r-1)}{2^{k+r}r!}={}^{k+r-1}C_r\left(\frac{1}{2}\right)^{k+r}$ for $r=0,1,2,...$

(Alternatively, $P(Y=r)$ is coefficient of t^r in $G_Y(t)$ which can be expanded binomially to yield the same result.)

13.
(i) $F(x)=P(X<x)=P(\cos\theta<x)=P(\cos^{-1}x<\theta<2\pi-\cos^{-1}x)$
Therefore, $F(x)=\frac{2\pi-2\cos^{-1}x}{2\pi}$
So $(x)=\frac{dF}{dx}=\frac{1}{\pi\sqrt{1-x^2}}$, for $-1\le x\le 1$

$E(X)=0$
$E(X^2)=\int_{-1}^{1}x^2\frac{1}{\pi\sqrt{1-x^2}}\quad dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 u}{\pi}du=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 2u}{2\pi}\ du=\frac{1}{2}$
So $Var(X)=\frac{1}{2}$
If $X=x$, $Y=\pm\sqrt{1-x^2}$ equiprobably, so $E(XY)=0$, $E(Y)=0$ and thus $Cov(X,Y)=0$, and hence $Corr(X,Y)=0$.

X and Y are not independent for if $X = x$, $Y = \pm\sqrt{1 - x^2}$ only, whereas without the restriction, Y can take all values in $[-1,1]$.

(ii) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 0$, and $E(\bar{Y}) = 0$ similarly.

$E(\bar{X}\bar{Y}) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{j=i}^{n} Y_j\right) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right)$ as X_i, Y_j are independent and each have expectation zero.

$E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right) = 0$ from part (i), and so $E(\bar{X}\bar{Y}) = 0$. Thus $Cov(\bar{X}, \bar{Y}) = 0$, and hence $Corr(\bar{X}, \bar{Y}) = 0$ as required.

For large n , $\bar{X} \sim N\left(0, \frac{1}{2n}\right)$ approximately, by Central Limit Theorem.

Thus,

$$P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx P\left(|z| \leq \frac{\sqrt{\frac{2}{n}}}{\frac{1}{\sqrt{2n}}}\right) = P(|z| \leq 2) \approx P(|z| \leq 1 \cdot 960) \approx 0 \cdot 95$$